

Correlation Functions of Complex Matrix Models

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Abstract

For a restricted class of potentials (harmonic+Gaussian potentials), we express the resolvent integral for the correlation functions of simple traces of powers of complex matrices of size N , in term of a determinant; this determinant is function of four kernels constructed from the orthogonal polynomials corresponding to the potential and from their Cauchy transform. The correlation functions are a sum of expressions attached to a set of fully packed oriented loops configurations; for rotational invariant systems, explicit expressions can be written for each configuration and more specifically for the Gaussian potential, we obtain the large N expansion ('t Hooft expansion) and the so-called BMN limit.

1 Introduction

Given a potential $V(M, M^+)$ where M is a $N \times N$ matrix with complex elements, we define the partition function as

$$Z = \int dM dM^+ e^{-Tr V(M, M^+)} \quad (1.1)$$

where we integrate over all matrix elements. We define the correlation functions as

$$\langle \prod_{i=1}^n O_i(M, M^+) \rangle = \frac{1}{Z} \int dM dM^+ \prod_{i=1}^n O_i(M, M^+) e^{-Tr V(M, M^+)} \quad (1.2)$$

Most observables $O(M, M^+)$ can be expressed in terms of traces of product of the matrices M and M^+ ; among these traces, we distinguish between "mixed traces" like $Tr [M^{J_1} (M^+)^{K_1} M^{J_2} (M^+)^{K_2} \dots]$ which are also called "words",

and simple traces where there is only one type of matrix (M or M^+) inside the trace. In this publication, we restrict ourselves to the case of simple traces

$$< \prod_{i=1}^p [Tr M^{J_i}] \prod_{i=1}^q [Tr (M^+)^{K_i}] > \quad J_i \text{ and } K_i > 0 \quad (1.3)$$

for a restricted class of potentials (described below). We now explain the method used to obtain the above correlation functions.

If certain conditions on the potential $V(M, M^+)$ are satisfied, we were able in Ref.[1] to calculate the following expectation values

$$\frac{1}{Z} \int dM dM^+ \frac{\prod_{i=1}^{L_2} \det(\eta_i^* - M^+)}{\prod_{i=1}^{M_2} \det(x_i^* - M^+)} \frac{\prod_{i=1}^{L_1} \det(\xi_i - M)}{\prod_{i=1}^{M_1} \det(y_i - M)} e^{-Tr V(M, M^+)} \quad (1.4)$$

for any numbers L_1, M_1, L_2, M_2 of determinants. The general result is essentially a determinant constructed from four kernels, each kernel is a function of the external sources $\xi_i, \eta_i^*, y_i, x_i^*$ which enter the orthogonal polynomials relative to the potential $V(z, z^*)$ and their Cauchy transform. The structure of this determinant is very dependant of the relative values of L_1, M_1, L_2, M_2 . In this publication we are only interested in the special case where $L_1 = M_1, L_2 = M_2$

since in that case, if we perform the derivatives $\prod_{i=1}^{M_2} \left(\frac{\partial}{\partial \eta_i^*} \right)_{x_i^* = \eta_i^*} \prod_{i=1}^{M_1} \left(\frac{\partial}{\partial \xi_i} \right)_{y_i = \xi_i}$ on (1.4) we obtain

$$\frac{1}{Z} \int dM dM^+ \prod_{i=1}^{M_2} Tr \frac{1}{\eta_i^* - M^+} \prod_{i=1}^{M_1} Tr \frac{1}{\xi_i - M} e^{-Tr V(M, M^+)} \quad (1.5)$$

The above resolvent integrals are the generating functionals for the integrals

$$\frac{1}{Z} \int dM dM^+ \prod_{i=1}^{M_2} Tr [(M^+)^{K_i}] \prod_{i=1}^{M_1} Tr [(M)^{J_i}] e^{-Tr V(M, M^+)} \quad (1.6)$$

which describe the correlation functions (1.3).

Let us now describe the conditions required by the potential $V(M, M^+)$. Any matrix M can be triangularized by a unitary matrix

$$M = U [D + T] U^+ \quad (1.7)$$

where D is the diagonal matrix of the eigenvalues (z_1, \dots, z_N) of M , and T is an upper strictly triangular matrix. Consequently, any power of M and M^+ is of

the form

$$M^k = U \left[D^k + T_k \right] U^+ \quad (1.8.a)$$

$$(M^+)^k = U \left[(D^+)^k + T_k^+ \right] U^+ \quad (1.8.b)$$

where T_k is an upper triangular matrix which also depends of the eigenvalues z_i for $k > 1$. Clearly, we have

$$Tr M^k = \sum_{i=1}^N z_i^k \quad (1.9.a)$$

$$Tr (M^+)^k = \sum_{i=1}^N (z_i^*)^k \quad (1.9.b)$$

On the other hand,

$$Tr MM^+ = \sum_{i=1}^N z_i z_i^* + Tr TT^+ \quad (1.10)$$

In the transformation (1.7),

$$\det (\xi_i - M) = \det (\xi_i - D) \quad (1.11.a)$$

$$\det (\eta_i^* - M^+) = \det (\eta_i^* - D^+) \quad (1.11.b)$$

and the integration measure becomes

$$dM dM^+ = dU dT dT^+ dD dD^+ \prod_{i < j} |z_i - z_j|^2 \quad (1.12)$$

We note that $dM dM^+$ represents $2N^2$ variables of integration while dU represents $N(N-1)$ variables of integration, $dT dT^+$ represents $N(N-1)$ variables of integration and $dD dD^+$ represents $2N$ variables of integration. The Jacobian of the transformation is given by the product of two Vandermonde determinants $\Delta(z) \Delta(z^*)$ where

$$\Delta(z) = \prod_{i < j} (z_i - z_j) = \begin{vmatrix} \pi_{N-1}(z_1) & \dots & \pi_{N-1}(z_N) \\ \dots & \dots & \dots \\ \pi_0(z_1) & \dots & \pi_0(z_N) \end{vmatrix} \quad (1.13)$$

and where the polynomials $\{\pi_0(z), \pi_1(z), \pi_2(z), \dots, \pi_{N-1}(z)\}$ are any set of monic polynomials of successive degree $0, 1, 2, \dots, N-1$.

Consequently, if we choose the potential $V(M, M^+)$ to be of the form

$$V(M, M^+) = MM^+ + V(M) + \overline{V}(M^+) \quad (1.14)$$

we may separate the integrations over the unitary group dU and over the upper triangular part $dT dT^+ e^{-Tr TT^+}$ and these two contributions cancel in the ratio of the numerator and the denominator of (1.6). We are left with integrals over the eigenvalues only.

More generally, we consider the integrals Z_N , I_N and J_N and a potential $V(z, z^*)$ such that the following integrals exist

$$Z_N = \int d\mu(z, z^*) \quad (1.15.a)$$

$$I_N = \frac{1}{Z_N} \int d\mu(z, z^*) \prod_{j=1}^N \left\{ \frac{\prod_{i=1}^{L_2} (z_j^* - \eta_i^*)}{\prod_{i=1}^{M_2} (z_j^* - x_i^*)} \frac{\prod_{i=1}^{L_1} (z_j - \xi_i)}{\prod_{i=1}^{M_1} (z_j - y_i)} \right\} \quad (1.15.b)$$

$$J_N = \frac{1}{Z_N} \int d\mu(z, z^*) \prod_{i=1}^{M_2} \left\{ \sum_{j=1}^N \frac{1}{z_j^* - \eta_i^*} \right\} \prod_{i=1}^{M_1} \left\{ \sum_{j=1}^N \frac{1}{z_i - \xi_i} \right\} \quad (1.15.c)$$

$$d\mu(z, z^*) = \prod_{i=1}^N dz_i dz_i^* \prod_{i < j} |z_i - z_j|^2 e^{-\sum_{i=1}^N V(z_i, z_i^*)} \quad (1.15.d)$$

The integrals J_N are generating functionals for the integrals

$$\frac{1}{Z_N} \int d\mu(z, z^*) \prod_{i=1}^{M_2} \left\{ \sum_{j=1}^N (z_j^*)^{K_i} \right\} \prod_{i=1}^{M_1} \left\{ \sum_{j=1}^N (z_j)^{J_i} \right\} \quad (1.16)$$

A second condition on $V(M, M^+)$ is that $V(z, z^*)$ is real and admits a set of orthogonal polynomials so that the technique of Ref[1] can be applied (the reality of $V(z, z^*)$ could be forgotten at the price of introducing biorthogonal polynomials but we shall not consider this case). We introduce the infinite set $\{p_n(z)\}$ of orthogonal, monic polynomials such that

$$\int d^2 z p_m^*(z) p_n(z) e^{-V(z, z^*)} = h_n \delta_{nm} \quad (1.17)$$

where $p_m^*(z)$ is a short notation for $[p_m(z)]^*$. Then, we consider the measure

$$d\mu(z, z^*; \xi_i, \eta_i^*; y_i, x_i^*) = d^2 z \frac{\prod_{i=1}^{L_2} (z^* - \eta_i^*)}{\prod_{i=1}^{M_2} (z^* - x_i^*)} \frac{\prod_{i=1}^{L_1} (z - \xi_i)}{\prod_{i=1}^{M_1} (z - y_i)} e^{-V(z, z^*)} \quad (1.18)$$

We proved in Ref.[1] the existence and we constructed explicitly the set of monic, biorthogonal polynomials $q_n(z; \xi_i, \eta_i^*; y_i, x_i^*)$ and $q_n^*(z; \eta_i, \xi_i^*; x_i, y_i^*)$ satisfying

$$\int d\mu(z, z^*; \xi_i, \eta_i^*; y_i, x_i^*) q_m^*(z; \eta_i, \xi_i^*; x_i, y_i^*) q_n(z; \xi_i, \eta_i^*; y_i, x_i^*) = \|q_n\|^2 \delta_{nm} \quad (1.19)$$

This construction is a generalization of Christoffel's result [2] which shows that given a positive Borel measure of one variable $d\mu(x)$ on the real line and its infinite set of orthogonal polynomials, it is possible to construct an infinite set of orthogonal polynomials for the measure

$$d\mu(x; \xi_i) = d\mu(x) \prod_{i=1}^L (x - \xi_i) \quad (1.20)$$

This result was extended to measures with external sources at the denominator by Uvarov [3] in 1969 and recently by Fyodorov and Strahov [4]. In 2003, Akemann and Vernizzi extended Christoffel's result to measures on the complex plane with external sources at the numerator [5] (see also Ref.[6]); their work was generalized to sources at the denominator as well in Ref.[1] and [7].

The consequence of the existence of biorthogonal polynomials for the measure (1.18) is that

$$Z_N = N! \prod_{i=0}^{N-1} h_i \quad (1.21.a)$$

$$Z_N I_N = N! \prod_{i=0}^{N-1} \|q_i\|^2 \quad (1.21.b)$$

In the case $L_1 = M_1$, $L_2 = M_2$, the pseudonorms $\|q_i\|^2$ are found in Ref.[1] to be the ratio of two determinants

$$\|q_i\|^2 = h_i \frac{D_i}{D_{i-1}} \quad i \geq 0 \quad (1.22)$$

where D_n is the determinant

$$\begin{vmatrix} N_n(\xi_1, y_1) & \dots & N_n(\xi_{M_1}, y_1) & A_n(x_1^*, y_1) & \dots & A_n(x_{M_2}^*, y_1) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ N_n(\xi_1, y_{M_1}) & \dots & N_n(\xi_{M_1}, y_{M_1}) & A_n(x_1^*, y_{M_1}) & \dots & A_n(x_{M_2}^*, y_{M_1}) \\ K_n(\xi_1, \eta_1^*) & \dots & K_n(\xi_{M_1}, \eta_1^*) & N_n^*(\eta_1, x_1) & \dots & N_n^*(\eta_1, x_{M_2}) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ K_n(\xi_1, \eta_{M_2}^*) & \dots & K_n(\xi_{M_1}, \eta_{M_2}^*) & N_n^*(\eta_{M_2}, x_1) & \dots & N_n^*(\eta_{M_2}, x_{M_2}) \end{vmatrix} \quad (1.23)$$

and D_{-1} is defined in (1.32). Consequently, in the case $L_1 = M_1$, $L_2 = M_2$ the integral I_N is found to be

$$I_N = (-)^{\frac{M_1(M_1-1)}{2}} (-)^{\frac{M_2(M_2-1)}{2}} \frac{\prod_{i,j=1}^{M_2} (x_i^* - \eta_j^*) \prod_{i,j=1}^{M_1} (y_i - \xi_j)}{\Delta(x^*) \Delta(y) \Delta(\eta^*) \Delta(\xi)} D_{N-1} \quad (1.24)$$

Then, by application of the derivatives $\prod_{i=1}^{M_2} \left(\frac{\partial}{\partial \eta_i^*} \right)_{x_i^* = \eta_i^*} \prod_{i=1}^{M_1} \left(\frac{\partial}{\partial \xi_i} \right)_{y_i = \xi_i}$ on (1.24), we proved in Ref.[1] that

$$J_N = "D_{N-1}" \quad (1.25)$$

where " D_n " is the "subtracted determinant"

$$\begin{vmatrix} H_n(\xi_1, \xi_1) & \dots & N_n(\xi_{M_1}, \xi_1) & A_n(\eta_1^*, \xi_1) & \dots & A_n(\eta_{M_2}^*, \xi_1) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ N_n(\xi_1, \xi_{M_1}) & \dots & H_n(\xi_{M_1}, \xi_{M_1}) & A_n(\eta_1^*, \xi_{M_1}) & \dots & A_n(\eta_{M_2}^*, \xi_{M_1}) \\ K_n(\xi_1, \eta_1^*) & \dots & K_n(\xi_{M_1}, \eta_1^*) & H_n^*(\eta_1, \eta_1) & \dots & N_n^*(\eta_{M_2}, \eta_1) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ K_n(\xi_1, \eta_{M_2}^*) & \dots & K_n(\xi_{M_1}, \eta_{M_2}^*) & N_n^*(\eta_1, \eta_{M_2}) & \dots & H_n^*(\eta_{M_2}, \eta_{M_2}) \end{vmatrix} \quad (1.26)$$

The determinant D_n is expressed in terms of four kernels K_n , N_n , N_n^* , A_n which are defined with some of their properties at the end of this introduction. The determinant " D_n " is obtained from the determinant D_n by changing all kernels N_n on the diagonal into H_n , changing all y_k into ξ_k , all x_k^* into η_k^* and finally by ignoring all double poles at $\xi_j = \xi_k$ and at $\eta_j^* = \eta_k^*$ (this operation is denoted by " \rightarrow ") as we develop the determinant (1.26) with terms like $N_n(\xi_j, \xi_k)$ $N_n(\xi_k, \xi_j)$ or $N_n^*(\eta_k, \eta_j)$ $N_n^*(\eta_j, \eta_k)$. In fact, from (1.30.b) we observe that there is no single poles either at $\xi_j = \xi_k$ or at $\eta_j^* = \eta_k^*$ since the residues are zero

$$"N_n(\xi_j, \xi_k) N_n(\xi_k, \xi_j)" = H_n(\xi_j, \xi_k) H_n(\xi_k, \xi_j) + \frac{H_n(\xi_j, \xi_k) - H_n(\xi_k, \xi_j)}{\xi_j - \xi_k} \quad (1.27)$$

In section 2, we develop the determinant " D_n " in terms of fully packed oriented loops configurations. Then, in section 3, we give an expression for each configuration in the case of rotationnally invariant systems ; finally in section 4, we use the Gaussian potential to obtain the correlation functions (1.3) which in that case are nothing but the number of graphs obtained from (1.6) by Wick's contraction. The large N expansion (where N is the size of the matrix) is considered and provides the number of graphs according to their genus (t'Hooft expansion [8]); another limit is also considered, namely the so-called BMN limit [9], where J_i , K_i in (1.3) and N are large but the ratios $\frac{J_i}{\sqrt{N}}$ and $\frac{K_i}{\sqrt{N}}$ are constant. Similar results were presented in [10] and [11].

The end of this introduction is devoted to the description of the kernels K_n , N_n , A_n and to some of their properties. We refer to Ref.[1] for the technical details; here, we simply give the definition of these kernels

$$K_n(\xi, \eta^*) = \sum_{i=0}^n \frac{p_i(\xi) p_i^*(\eta)}{h_i} \quad (1.28)$$

We introduce the following integrals (in the case of potentials of one variable, the integral (1.29.a) is called the Cauchy-Hilbert transform of the corresponding polynomial)

$$t_n(y) = \int d^2 z \, p_n^*(z) \frac{1}{z-y} e^{-V(z, z^*)} \quad (1.29.a)$$

$$Q(x^*, y) = \int d^2 z \, \frac{1}{(z^* - x^*)(z - y)} e^{-V(z, z^*)} \quad (1.29.b)$$

then,

$$H_n(\xi, y) = \sum_{i=0}^n \frac{p_i(\xi) t_i(y)}{h_i} \quad (1.30.a)$$

$$N_n(\xi, y) = \frac{1}{y - \xi} + H_n(\xi, y) \quad (1.30.b)$$

$$A_n(x^*, y) = \sum_{i=0}^n \frac{t_i^*(x) t_i(y)}{h_i} - Q(x^*, y) \quad (1.30.c)$$

In (1.22), we also use the extensions

$$K_{-1}(\xi, \eta^*) = 0 \quad (1.31.a)$$

$$N_{-1}(\xi, y) = \frac{1}{y - \xi} \quad (1.31.b)$$

$$A_{-1}(x^*, y) = -Q(x^*, y) \quad (1.31.c)$$

and the determinant

$$D_{-1} = \begin{vmatrix} N_{-1}(\xi_j, y_i) & A_{-1}(x_l^*, y_i) \\ 0 & N_{-1}^*(\eta_k, x_l) \end{vmatrix} \quad (1.32.a)$$

$$D_{-1} = |N_{-1}(\xi_j, y_i)| \cdot |N_{-1}^*(\eta_k, x_l)| \quad (1.32.b)$$

where

$$|N_{-1}(\xi_j, y_i)| = \begin{vmatrix} \frac{1}{y_1 - \xi_1} & \cdots & \frac{1}{y_1 - \xi_{M_1}} \\ \vdots & \ddots & \vdots \\ \frac{1}{y_{M_1} - \xi_1} & \cdots & \frac{1}{y_{M_1} - \xi_{M_1}} \end{vmatrix} \quad (1.33.a)$$

$$|N_{-1}(\xi_j, y_i)| = (-1)^{\frac{M_1(M_1-1)}{2}} \frac{\Delta(y) \Delta(\xi)}{\prod_{i,j} (y_i - \xi_j)} \quad (1.33.b)$$

In Ref.[1], we prove the following properties

$$\sum_{i=0}^{\infty} \frac{p_i(\xi) t_i(y)}{h_i} = \frac{1}{\xi - y} \quad (1.34.a)$$

$$\sum_{i=0}^{\infty} \frac{t_i^*(x) t_i(y)}{h_i} = Q(x^*, y) \quad (1.34.b)$$

so that we have the formal power series

$$N_n(\xi, y) = - \sum_{i=n+1}^{\infty} \frac{p_i(\xi) t_i(y)}{h_i} \quad (1.35.a)$$

$$A_n(x^*, y) = - \sum_{i=n+1}^{\infty} \frac{t_i^*(x) t_i(y)}{h_i} \quad (1.35.b)$$

Finally, we have the asymptotic behaviour

$$t_n(y) \sim - \frac{h_n}{y^{n+1}} \quad \text{as } y \rightarrow \infty \quad (1.36.a)$$

$$N_n(\xi, y) \sim \frac{p_{n+1}(\xi)}{y^{n+2}} \quad \text{as } y \rightarrow \infty \quad (1.36.b)$$

$$A_n(x^*, y) \sim \frac{t_{n+1}^*(x)}{y^{n+2}} \quad \text{as } y \rightarrow \infty \quad (1.36.c)$$

2 The graph description of the determinant " D_n "

The determinant " D_n " as given in (1.26) is a sum of products of oriented propagators joining the various points (for simplicity, we label the points and the corresponding variables by the same letter). Let us split the plane into two parts separated by a border line; on the right part we draw the various points η_i^* and in the left part we draw the various points ξ_i . As we develop the determinant, every time we meet a kernel $A_n(\eta_i^*, \xi_j)$ we draw an oriented propagator from the point η_i^* towards the point ξ_j crossing the border from right to left; similarly with the kernel $K_n(\xi_i, \eta_j^*)$ we draw an oriented propagator from the point ξ_i towards the point η_j^* crossing the border from left to right. Then, inside the ξ -region we draw an oriented propagator from the point ξ_i towards the point ξ_j corresponding to the kernel $N_n(\xi_i, \xi_j)$; in addition, the diagonal part of " D_n " defines self closing loops on one point ξ_i from the kernel $H_n(\xi_i, \xi_i)$. Similarly, inside the η^* -region we draw an oriented propagator from the point η_i^* towards the point η_j^* corresponding to the kernel $N_n^*(\eta_j, \eta_i)$; in addition, the diagonal part of " D_n " defines self closing loops on one point η_i^* from the kernel $H_n^*(\eta_i, \eta_i)$.

As we develop the determinant, the product of propagators describes a number of disjoint oriented closed loops γ_a (including self-loop with one point) so that any point ξ_i or η_i^* belongs to one closed loop γ_a and only one; we call such a configuration "fully packed oriented loops configuration" which we denote by ϖ (fig.1); the number of oriented loops in ϖ is denoted $\#(\varpi)$

$$\varpi = \cup_{a=1}^{\#(\varpi)} \gamma_a \quad (2.1)$$

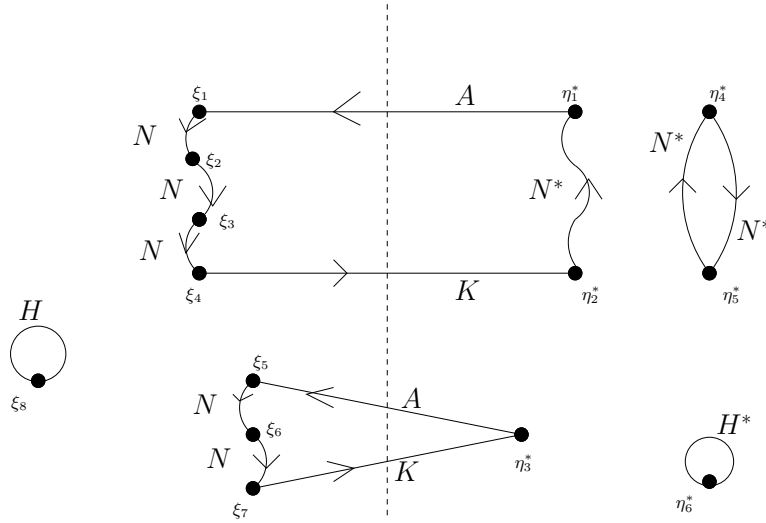


fig.1 : contribution to $W_{8,6}(\xi_1, \dots, \xi_8; \eta_1^*, \dots, \eta_6^*)$

The determinant " D_n " is the sum over the $(M_1 + M_2)!$ configurations ϖ

$$D_n = (-1)^{M_1+M_2} \sum_{\varpi} \prod_{a=1}^{\#(\varpi)} (-I_{\gamma_a}) \quad (2.2)$$

Given an oriented loop γ , which is not made of N propagators alone and/or of N^* propagators alone, the amplitude I_γ is the product of a certain number $q > 0$ of propagators K_n and of the same number q of propagators A_n ; the remaining propagators describe q chains of propagators N and q chains of propagators N^* (such chains may be reduced to a single point). We call skeleton graph an oriented loop γ without N and N^* propagators. To any oriented loop γ we associate a skeleton graph by contraction of each of its chains of propagators N or N^* into a point.

Along a given chain, the propagators N or N^* have poles when the ξ 's or the η 's coincide. However, these poles cancel when we sum over all permutations of the ξ 's or of the η 's inside a given chain of propagators N or N^* . In appendix

We show how the poles cancel when we sum over several configurations which have the same associated skeleton graph.

For a given potential $V(z, z^*)$, we must calculate the orthogonal polynomials $p_n(z)$ and their Cauchy transform $t_n(z)$ in order to evaluate the kernels which are the propagators of the graphs. Of course, each case is a specific case; however, some general structure can be obtained in the special case where the system (potential and domain of integration) is rotationnally invariant.

3 The $\frac{1}{\xi}, \frac{1}{\eta^*}$ expansion in the case of the rotationnally invariant systems

The expressions for the four kernels K_n, N_n, N_n^*, A_n simplify greatly in the case of rotationnally invariant systems because the angular integrations in (1.17) is trivially performed. Given a potential $V(zz^*)$ and a disc of radius R which might be the complex plane ($R = \infty$) if $\exp(-V(zz^*))$ decreases strongly enough at ∞ . In that case, the orthogonal polynomials are the so-called Ginibre's polynomials

$$p_n(z) = z^n \quad (3.1)$$

and their norms squared are

$$h_n = 2\pi \int_0^R d\rho \rho^{2n+1} e^{-V(\rho^2)} \quad (3.2)$$

In the Gaussian case where $V(zz^*) = zz^*$ and $R = \infty$,

$$h_n = \pi \Gamma(n+1) \quad (3.3)$$

The function $t_n(y)$ defined in (1.29.a) is found to be

$$t_n(y) = -\frac{h_n}{y^{n+1}}, \quad |y| > R \quad (3.4)$$

if $R = \infty$ this result is still valid up to exponentially small terms when $y \rightarrow \infty$ and can be used as a formal power series (see also (1.34.a)).

According to (1.28) and (1.30) the formal power series for the kernels K_n, H_n, N_n and A_n are

$$K_n(\xi, \eta^*) = \sum_{j=0}^n \frac{(\xi \eta^*)^j}{h_j} \quad (3.5.a)$$

$$H_n(\xi_i, \xi_j) = -\frac{1}{\xi_j^{n+1}} \frac{\xi_i^{n+1} - \xi_j^{n+1}}{\xi_i - \xi_j} \quad \xi_i \neq \xi_j \quad (3.5.b)$$

so that

$$H_n(\xi_i, \xi_i) = -\frac{n+1}{\xi_i} \quad (3.6.a)$$

$$N_n(\xi_i, \xi_j) = \left(\frac{\xi_i}{\xi_j}\right)^{n+1} \frac{1}{\xi_j - \xi_i} \quad (3.6.b)$$

Finally,

$$A_n(\eta^*, \xi) = -\sum_{i=n+1}^{\infty} \frac{h_i}{(\xi\eta^*)^{i+1}} \quad (3.7)$$

It is now possible to calculate for any closed loop γ the corresponding amplitude I_γ which contributes to the value of the determinant " D_n ".

1°) the $\frac{1}{\xi}, \frac{1}{\eta^*}$ expansion for the skeleton graphs: given a skeleton graph with $q > 0$ propagators A and q propagators K , the expression for I_γ can be written

$$I_\gamma = \prod_{k=1}^q A_n(\eta_{\alpha_k}^*, \xi_{\alpha_k}) \prod_{i=1}^q K_n(\xi_{\alpha_k}, \eta_{\alpha_{k+1}}^*) \quad , \text{ with } \eta_{\alpha_{q+1}}^* = \eta_{\alpha_1}^* \quad (3.8)$$

As we expand the propagators A_n and K_n according to (3.5.a) and (3.7) we introduce an internal momentum i_k for $A_n(\eta_{\alpha_k}^*, \xi_{\alpha_k})$ and an internal momentum j_k for $K_n(\xi_{\alpha_k}, \eta_{\alpha_{k+1}}^*)$; the summation over i_k runs from $n+1$ to ∞ while the summation over j_k runs from 0 to n . By convention, we extend the definition of h_j to negative j 's as $h_{j<0} = \infty$ so that in K_n we may eventually ignore the lower limit of summation.

The expansion of I_γ is of the form

$$I_\gamma = \sum_{\{J's > 0, K's > 0\}} \prod_{k=1}^q \left(\frac{1}{\xi_{\alpha_k}^{J_k+1}}\right) \prod_{k=1}^q \left(\frac{1}{\eta_{\alpha_k}^{K_k+1}}\right)^* I(J_k, K_k) \quad (3.9)$$

where

$$I(J_k, K_k) = (-)^q \sum_{D_{\{J_k, K_k\}}} \frac{h_{i_1} \dots h_{i_q}}{h_{j_1} \dots h_{j_q}} \quad (3.10)$$

and $D_{\{J_k, K_k\}}$ is the summation domain for the indices i_k and j_k . The determination of this domain is the main difficulty. Clearly, all the indices i 's and j 's are fixed in terms of the J 's and the K 's up to a translation.

We first describe some simple exemples:

a) let us consider the graph γ_1 (fig.2).

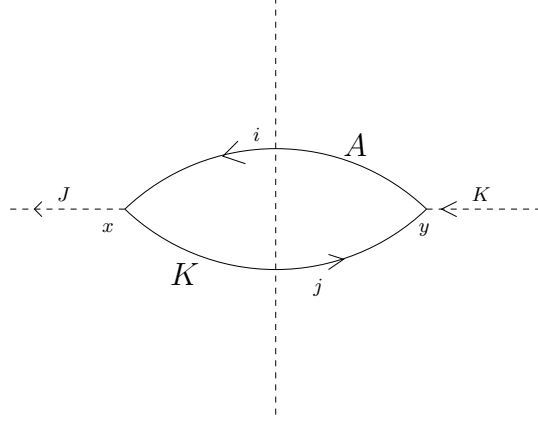


fig. 2 : graph $\gamma_1 \rightarrow W_{1,1}^c(x, y)$

In that case, the momentum conservation law is

$$i - j = J = K \quad (3.11)$$

so that the domain of summation for the momentum j is

$$n - J + 1 \leq j \leq n \quad (3.12)$$

Consequently,

$$I_{\gamma_1}(J, K) = -\delta_{J,K} \sum_{j=n-J+1}^n \frac{h_{j+J}}{h_j} \quad (3.13)$$

The equation (3.13) is still valid in the case where $n - J + 1 < 0$ with the convention $h_{j<0} = \infty$. We define the characteristic function for the skeleton graph γ_1

$$\Phi_{\gamma_1}(n, J) = \sum_{j=0}^n \frac{h_{j+J}}{h_j} \quad (3.14)$$

then,

$$I_{\gamma_1}(J, K) = -\delta_{J,K} (1 - T_J) \Phi_{\gamma_1}(n, J) \quad (3.15)$$

where the operator T_J is a translation operator over n

$$T_J f(n) = f(n - J) \quad (3.16)$$

b) let us consider the graph γ_2 (fig.3).

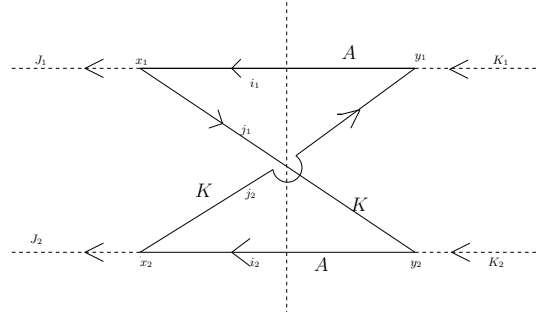


fig. 3 : graph $\gamma_2 \rightarrow$ contribution to $W_{2,2}^c(x_1, x_2; y_1, y_2)$

Choosing $j_2 = j$ as the independant loop momentum, we observe the following constraints

$$j \geq n + 1 - K_1, \quad j \geq n + 1 - J_2 \quad (3.17.a)$$

$$j \leq n - K_1 + J_1, \quad j \leq n \quad (3.17.b)$$

$$J = J_1 + J_2 = K_1 + K_2 = K \quad (3.17.c)$$

Consequently, we obtain

$$I_{\gamma_2}(J_1, J_2, K_1, K_2) = \delta_{J,K} O(J_i, K_i) \Phi_{\gamma_2}(n, J_i, K_i) \quad (3.18.a)$$

$$O(J_i, K_i) = T_{Sup(K_1 - J_1, 0)} - T_{Inf(K_1, J_2)} \quad (3.18.b)$$

$$\Phi_{\gamma_2}(n, J_i, K_i) = \sum_{j=0}^n \frac{h_{j+K_1}}{h_{j+K_1-J_1}} \frac{h_{j+J_2}}{h_j} \quad (3.18.c)$$

c) the generalization to the skeleton graph γ_q with q pairs of propagators A_n and K_n is now straightforward. We define

$$J = \sum_{k=1}^q J_k \quad K = \sum_{k=1}^q K_k \quad (3.19)$$

and we choose $j = j_q$ as independant loop momentum so that

$$i_k = j_q + b_k \quad j_k = j_q + a_k \quad (3.20)$$

with

$$a_k = \sum_{p=1}^k (K_p - J_p), \quad a_{q=0} \quad (3.21.a)$$

$$b_k = a_k + J_k \geq 1 \quad (3.21.b)$$

We note from (3.20) that any variable b_k is strictly larger than any variable $a_{k'}$. The contribution of the graph γ_q is

$$I_{\gamma_q}(J_i, K_i) = (-)^q \delta_{J,K} (T_X - T_Y) \Phi_{\gamma_q}(n, J_i, K_i) \quad (3.22.a)$$

$$\Phi_{\gamma_q}(n, J_i, K_i) = \sum_{j=0}^n \frac{h_{j+b_1} \dots h_{j+b_q}}{h_{j+a_1} \dots h_{j+a_q}} \quad (3.22.b)$$

with

$$0 \leq X = \text{Sup}(a_k) \quad (3.23.a)$$

$$1 \leq Y = \text{Inf}(b_k) \quad (3.23.b)$$

2°) the $\frac{1}{\xi}, \frac{1}{\eta}$ expansion for any graph: a one loop graph γ is made of q propagators A , q propagators K , q chains (C_1, \dots, C_q) of propagators N and q chains (C_1^*, \dots, C_q^*) of propagators N^* (the chains can be reduced to a single point). The case $q = 0$ is made of loops of N propagators alone, or of N^* propagators alone. To each chain C_i we associate an external outgoing momentum

$$\Delta_i = \sum_{\xi_k \in C_i} J_k \quad (3.24)$$

and to each chain C_i^* an external ingoing momentum

$$\Theta_i = \sum_{\eta_k \in C_i^*} K_k \quad (3.25)$$

Now, to each graph γ we associate a skeleton graph $s(\gamma)$ obtained from γ by shrinking into a point each of the chain C_i or C_i^* . To each of these points we associate the corresponding external momentum Δ_i or Θ_i ; we have

$$J = \sum_{i=1}^q \Delta_i = \sum_{i=1}^q \Theta_i = \sum_{\xi_k \in \gamma} J_k = \sum_{\eta_k \in \gamma} K_k = K \quad (3.26)$$

Because all J 's and K 's are positive or nul, equation (3.26) shows that if the graph γ is made of N propagators alone ($q = 0$), since the K 's are zero, the J 's are also zero. The only non zero contribution in this category is the self-closed loop with one point which is $H_n(\xi_i, \xi_i) = -\frac{n+1}{\xi_i}$. The two points loop is

$$N(\xi_i, \xi_j) N(\xi_j, \xi_i) = 0 \quad (3.27)$$

Larger loops of N propagators alone, or of N^* propagators alone are shown in appendix A to be zero.

The functions h_i are only present in the A and K propagators so that the function to be summed for a graph γ is the characteristic function of the skeleton graph $s(\gamma)$

$$\frac{h_{j+b_1} \dots h_{j+b_q}}{h_{j+a_1} \dots h_{j+a_q}} \quad (3.28)$$

where j is the momentum corresponding to the propagator K_q and

$$a_k = \sum_{p=1}^k (\Theta_p - \Delta_p), \quad a_{q=0} \quad (3.29.a)$$

$$b_k = a_k + \Delta_k \geq 1 \quad (3.29.b)$$

The equation (3.10) for a one loop graph γ with propagators N and N^* can be written

$$I_\gamma(J_i, K_i) = (-)^q \delta_{J,K} O(J_i, K_i) \sum_{j=0}^n \frac{h_{j+b_1} \dots h_{j+b_q}}{h_{j+a_1} \dots h_{j+a_q}} \quad (3.30)$$

where the operator $O(J_i, K_i)$ determines the range of summation for the variable j and is described below.

We introduce the commutative operation \otimes

$$T_a \otimes T_b = T_{Sup(a,b)} \quad (3.31.a)$$

$$1 \otimes T_b = T_{Sup(0,b)} \quad (3.31.b)$$

so that conditions like $n - a + 1 \leq j \leq n$ together with $n - b + 1 \leq j \leq n$, where a and b are positive, can be written

$$\sum_{j=n-Inf(a,b)+1}^n \dots = (1 - T_a) \otimes (1 - T_b) \sum_{j=0}^n \dots \quad (3.32)$$

Now, we proved in appendix A that for a given chain C_i with p points $((p-1)$ propagators), the sum over the $p!$ permutations generates a domain of summation for the variable j_i described by the operator $\prod_{\xi_r \in C_i} (1 - T_{J_r})$; since the propagator with momentum j_i is between the chains C_i and C_{i+1}^* (with $C_{q+1}^* = C_1^*$) the constraints on j_i can be written

$$\left[\prod_{\xi_r \in C_i} (1 - T_{J_r}) \right] \otimes \left[\prod_{\eta_s \in C_{i+1}^*} (1 - T_{K_s}) \right] \quad (3.33)$$

and using the relation (3.20) we can write the corresponding constraints on j_q as

$$T_{a_i} \left\{ \left[\prod_{\xi_r \in C_i} (1 - T_{J_r}) \right] \otimes \left[\prod_{\eta_s \in C_{i+1}^*} (1 - T_{K_s}) \right] \right\} \quad (3.34)$$

We collect all the constraints on the various momenta j_i for $i = 1, \dots, q$ and write

$$O(J_i, K_i) = \otimes_{i=1}^q \left[T_{a_i} \left\{ \left[\prod_{\xi_r \in C_i} (1 - T_{J_r}) \right] \otimes \left[\prod_{\eta_s \in C_{i+1}^*} (1 - T_{K_s}) \right] \right\} \right] \quad (3.35)$$

4 The Gaussian potential, large N and BMN expansion

We now apply the results of section 3 to the Gaussian potential with the function h_n given in (3.3); we also replace the index n of section 3 by $(N - 1)$ according to (1.25).

1°) let us consider the two points function

$$W_{1,1}(N, x, y) = \langle \text{Tr} \frac{1}{x - M} \text{Tr} \frac{1}{y - M^+} \rangle \quad (4.1)$$

which is equal from (1.26) to the determinant

$$D_{N-1} = \det \begin{vmatrix} H_{N-1}(x, x) & A_{N-1}(y, x) \\ K_{N-1}(x, y) & H_{N-1}(y, y) \end{vmatrix} \quad (4.2)$$

The part $H_{N-1}(x, x) H_{N-1}(y, y)$ of the determinant corresponds to a disconnected graph made of two self closing loops and from (3.6.a) is equal to $\frac{N^2}{xy}$. The part $A_{N-1}(y, x) K_{N-1}(x, y)$ of the determinant corresponds to the graph γ_1 (fig.2). The corresponding function $\Phi_{\gamma_1}(N - 1, J)$ is given in (3.14)

$$\Phi_{\gamma_1}(N - 1, J) = \sum_{j=0}^{N-1} \frac{\Gamma(j + J + 1)}{\Gamma(j + 1)} = \frac{1}{J + 1} \frac{\Gamma(N + J + 1)}{\Gamma(N)} \quad (4.3)$$

Consequently, according to (3.15)

$$A_{N-1}(y, x) K_{N-1}(x, y) = - \sum_{J=1}^{\infty} \frac{1}{(xy)^{J+1}} \frac{1}{J + 1} \left[\frac{\Gamma(N + J + 1)}{\Gamma(N)} - \frac{\Gamma(N + 1)}{\Gamma(N - J)} \right] \quad (4.4)$$

The square bracket [] in (4.4) has clearly a $\frac{1}{N^2}$ expansion

$$\frac{\Gamma(N + J + 1)}{\Gamma(N)} - \frac{\Gamma(N + 1)}{\Gamma(N - J)} = 2 N^J \sum_{p \text{ odd} > 0} \frac{1}{N^{p-1}} \sigma_p(J) \quad (4.5)$$

where $\sigma_p(J)$ is described in appendix B. After a convenient rescaling, we write $A_{N-1}(y\sqrt{N}, x\sqrt{N}) K_{N-1}(x\sqrt{N}, y\sqrt{N})$ as

$$- \frac{1}{N} \sum_{J=1}^{\infty} \frac{1}{(xy)^{J+1}} \left[J + \frac{1}{24N^2} (J + 1) J^2 (J - 1) (J - 2) + O\left(\frac{1}{N^4}\right) \right] \quad (4.6.a)$$

$$= - \frac{1}{N} \left[\frac{1}{(xy - 1)^2} + \frac{1}{N^2} \frac{xy(3xy + 2)}{(xy - 1)^6} + O\left(\frac{1}{N^4}\right) \right] \quad (4.6.b)$$

The so-called BMN limit of $I_{\gamma_1}(J, K)$ (defined in (3.15), (4.3)) is the large N , large J limit with constant $\frac{J^2}{N}, \frac{K^2}{N}$. From the large J behaviour of $\sigma_p(J)$ as given in appendix B, we obtain the BMN limit as

$$I_{\gamma_1}(J, K) \sim -\delta_{J,K} \frac{2 N^{J+1}}{J} sh\left(\frac{J^2}{2N}\right) \quad (4.7)$$

At this stage, we find convenient to introduce a formalism developped in appendix B and which makes the $\frac{1}{N^2}$ -expansion and the BMN limit transparant. We define

$$N^{**(J+1)} = \left(N - \frac{J}{2}\right) \left(N - \frac{J}{2} + 1\right) \dots \left(N + \frac{J}{2} - 1\right) \left(N + \frac{J}{2}\right) \quad (4.8)$$

which has manifestly a $\frac{1}{N^2}$ -expansion

$$N^{**(J+1)} = \sum_{p=0}^{E(\frac{J+1}{2})} (-)^p \Sigma_{2p}(J) N^{J-2p+1} \quad (4.9)$$

where the coefficients $\Sigma_{2p}(J)$ are given in appendix B. Consequently, the expression (4.4) can be written as

$$(1 - T_J) \frac{1}{J+1} \frac{\Gamma(N+J+1)}{\Gamma(N)} = \frac{2}{J+1} sh\left(\frac{J}{2} \frac{\partial}{\partial N}\right) N^{**(J+1)} \quad (4.10)$$

which is manifestly a $\frac{1}{N^2}$ -expansion. Also the BMN limit of $N^{**(J+1)}$ is simply N^{J+1} so that the BMN limit of (4.10) consists in replacing $\frac{\partial}{\partial N}$ by $\frac{J}{N}$ and $N^{**(J+1)}$ by N^{J+1} .

2°) we now consider the three points function

$$W_{2,1}(N, x_1, x_2, y) = < Tr \frac{1}{x_1 - M} Tr \frac{1}{x_2 - M} Tr \frac{1}{y - M^+} > \quad (4.11)$$

which is equal from (1.26) to the determinant

$$"D_{N-1}" = " \det " \begin{vmatrix} H_{N-1}(x_1, x_1) & N_{N-1}(x_2, x_1) & A_{N-1}(y, x_1) \\ N_{N-1}(x_1, x_2) & H_{N-1}(x_2, x_2) & A_{N-1}(y, x_2) \\ K_{N-1}(x_1, y) & K_{N-1}(x_2, y) & H_{N-1}(y, y) \end{vmatrix} \quad (4.12)$$

The connected contribution to (4.12) is

$$[A_{N-1}(y, x_1) N_{N-1}(x_1, x_2) K_{N-1}(x_2, y) + (x_1 \Leftrightarrow x_2)].$$

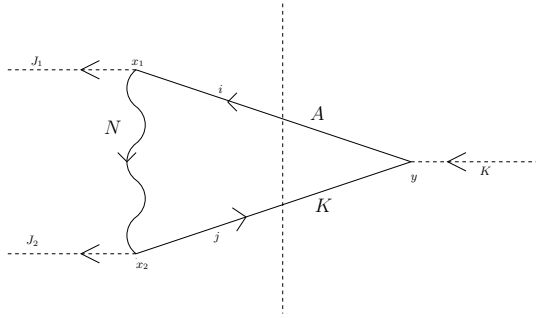


fig. 4 : contribution to $W_{2,1}^c(x_1, x_2; y)$

In section 3 and in appendix A, we prove that this combination is equal to

$$W_{2,1}^c(N, x_1, x_2, y) = \sum_{\{J_1, J_2, K\}=1}^{\infty} \frac{1}{x_1^{J_1+1}} \frac{1}{x_2^{J_2+1}} \frac{1}{y^{K+1}} I_2(N, J_1, J_2, K) \quad (4.13.a)$$

$$I_2(N, J_1, J_2, K) = -\delta_{J,K} (1 - T_{J_1})(1 - T_{J_2}) \frac{1}{J+1} \frac{\Gamma(N+J+1)}{\Gamma(N)} \quad (4.13.b)$$

$$I_2(N, J_1, J_2, K) = -\delta_{J,K} \frac{4}{J+1} sh\left(\frac{J_1}{2} \frac{\partial}{\partial N}\right) sh\left(\frac{J_2}{2} \frac{\partial}{\partial N}\right) N^{*(J+1)} \quad (4.13.c)$$

where $J = K = J_1 + J_2$. The large N behaviour of $W_{2,1}^c(N, x_1\sqrt{N}, x_2\sqrt{N}, y\sqrt{N})$ is found to be

$$\begin{aligned} & -\frac{y}{N^{\frac{5}{2}}} \sum_{\{J_1, J_2\}=1}^{\infty} \frac{1}{(yx_1)^{J_1+1}} \frac{1}{(yx_2)^{J_2+1}} J_1 J_2 J^* \\ & * \left[1 + \frac{1}{24N^2} (J-1)(J-2) \{(J+1)(J-2) - 2J_1 J_2\} + O\left(\frac{1}{N^4}\right) \right] \end{aligned} \quad (4.14)$$

or

$$W_{2,1}^c(N, x_1\sqrt{N}, x_2\sqrt{N}, y\sqrt{N}) = -\frac{2y}{N^{\frac{5}{2}}} \left[\frac{y^2 x_1 x_2 - 1}{(yx_1 - 1)^3 (yx_2 - 1)^3} + O\left(\frac{1}{N^2}\right) \right] \quad (4.15)$$

Finally from (4.13.c), the BMN limit, where N, J_1, J_2 are large with

$\frac{J_1^2}{N}, \frac{J_2^2}{N}$ constant, is obtained as

$$I_2(N, J_1, J_2, K) \sim -\delta_{J,K} \frac{4N^{J+1}}{J} sh\left(\frac{J_1 J}{2N}\right) sh\left(\frac{J_2 J}{2N}\right) \quad (4.16)$$

3°) the generalisation of the situation 2°) is the calculum of $W_{p,1}^c(N, x_1, \dots, x_p, y)$ and is easy to understand. We sum over $p!$ graphs obtained by exchanging the p points x_i and we calculate

$$W_{p,1}^c(N, x_i, y) = \left[A_{N-1}(y, x_1) \prod_{i=1}^{p-1} N_{N-1}(x_i, x_{i+1}) K_{N-1}(x_p, y) + \text{Sym}(x_i) \right] \quad (4.17)$$

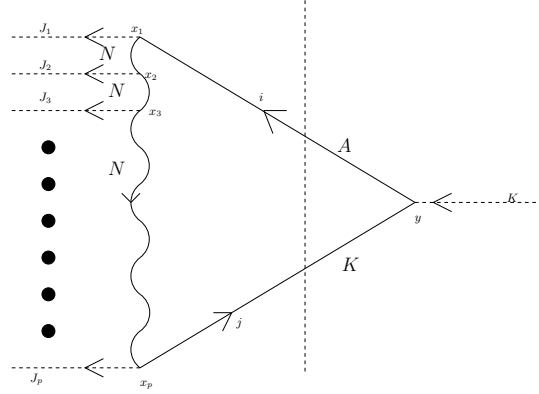


fig. 5 : contribution to $W_{p,1}^c(x_1, \dots, x_p; y)$

From section 3 and appendix A, we know that the poles of the propagators $N_{N-1}(x_i, x_{i+1})$ cancel in the sum and that the result can be written as

$$W_{p,1}^c(N, x_i, y) = \sum_{\{J_i, K\}=1}^{\infty} \prod_{i=1}^p \frac{1}{x_i^{J_i+1}} \frac{1}{y^{K+1}} I_p(N, J_i, K) \quad (4.18.a)$$

$$I_p(N, J_i, K) = -\delta_{J,K} \prod_{i=1}^p (1 - T_{J_i}) \frac{1}{J+1} \frac{\Gamma(N+J+1)}{\Gamma(N)} \quad (4.18.b)$$

$$I_p(N, J_i, K) = -\delta_{J,K} \frac{2^p}{J+1} \prod_{i=1}^p \text{sh} \left(\frac{J_i}{2} \frac{\partial}{\partial N} \right) N^{*(J+1)} \quad (4.18.c)$$

where $J = K = J_1 + \dots + J_p$. The asymptotic behaviour at large N of

$W(N, x_i \sqrt{N}, y \sqrt{N})$, after a proper rescaling of the variables, is

$$-\frac{y^{p-1}}{N^{\frac{3p-1}{2}}} \sum_{\{J_i\}=1}^{\infty} \prod_{i=1}^p \frac{1}{(x_i y)^{J_i+1}} \left[J_1 \dots J_p J (J-1) \dots (J-p+2) + O\left(\frac{1}{N^2}\right) \right] \quad (4.19)$$

or

$$W_{p,1}^c \left(N, x_i \sqrt{N}, y \sqrt{N} \right) = \frac{(-)^p}{N^{\frac{3p-1}{2}}} \left\{ \frac{\partial^{p-1}}{\partial y^{p-1}} \frac{y^{2p-2}}{\prod_{i=1}^p (x_i y - 1)^2} + O \left(\frac{1}{N^2} \right) \right\} \quad (4.20)$$

The BMN limit is trivially obtained as

$$I_p(N, J_i, K) \sim -\delta_{J,K} \frac{2^p N^{J+1}}{J} \prod_{i=1}^p sh \left(\frac{J_i J}{2N} \right) \quad (4.21)$$

4°) We now apply the formalism to the four points, connected, resolvent function $W_{2,2}^c(N, x_1, x_2, y_1, y_2)$. First we calculate the contribution of the graph of fig.6 and its crossed symmetric ones ($J_1 \Leftrightarrow J_2$ and $K_1 \Leftrightarrow K_2$).

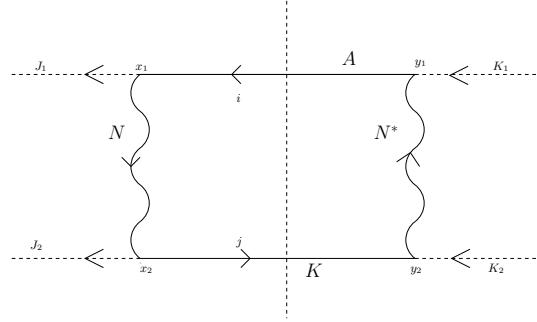


fig. 6 : contribution to $W_{2,2}^c(x_1, x_2; y_1, y_2)$

We write

$$W(N, x_1, x_2, y_1, y_2) = \sum_{\{J_i, K\}=1}^{\infty} \left(\prod_{i=1}^2 \frac{1}{x_i^{J_i+1}} \right) \left(\prod_{i=1}^2 \frac{1}{y_i^{K_i+1}} \right) I(N, J_i, K_i) \quad (4.22)$$

The formalism of section 3 and appendix A applied to the corresponding four graphs gives

$$I(N, J_i, K_i) = -\delta_{J,K} O(J_i, K_i) \frac{1}{J+1} \frac{\Gamma(N+J+1)}{\Gamma(N)} \quad (4.23.a)$$

$$O(J_i, K_i) = [(1 - T_{J_1}) (1 - T_{J_2}) \otimes (1 - T_{K_1}) (1 - T_{K_2})] \quad (4.23.b)$$

$$O(J_i, K_i) = (1 - T_{Inf}) (1 + T_{Sup}) \quad (4.23.c)$$

where the symbol \otimes is defined in (3.31) and where

$$J = K = J_1 + J_2 = K_1 + K_2 \quad (4.24.a)$$

$$Inf = Inf(J_i, K_i) \quad Sup = Sup(J_i, K_i) \quad (4.24.b)$$

The function $I(N, J_i, K_i)$ can also be written

$$I(N, J_i, K_i) = -\delta_{J,K} \frac{4}{J+1} sh\left(\frac{Inf}{2} \frac{\partial}{\partial N}\right) ch\left(\frac{Sup}{2} \frac{\partial}{\partial N}\right) N^{**(J+1)} \quad (4.25)$$

The BMN limit is immediately obtained as

$$I(N, J_i, K_i) \sim -\delta_{J,K} \frac{4N^{J+1}}{J} sh\left(\frac{J.Inf}{2N}\right) ch\left(\frac{J.Sup}{2N}\right) \quad (4.26)$$

The large N limit is obtained by expansion of the sh and ch in (4.25) as

$$I(N, J_i, K_i) = -\delta_{J,K} 2 Inf N^J * \left[1 + \frac{J(J-1)}{24N^2} [3 Sup^2 + Inf^2 - (J+2)] + O\left(\frac{1}{N^4}\right) \right] \quad (4.27)$$

5°) The remaining part of the connected four points resolvent function $W_{2,2}^c(N, x_1, x_2, y_1, y_2)$ is attached to the skeleton graphs of fig.3 and its crossed symmetric one ($J_1 \Leftrightarrow J_2$) or ($K_1 \Leftrightarrow K_2$). According to section 3, we have for the graph of fig.3

$$I_1(N, J_i, K_i) = \delta_{J,K} (T_X - T_Y) \sum_{j=0}^{N-1} \frac{\Gamma(j+K_1+1)}{\Gamma(j+K_1-J_1+1)} \frac{\Gamma(j+J_2+1)}{\Gamma(j+1)} \quad (4.28)$$

with $J = J_1 + J_2 = K_1 + K_2 = K$ and with

$$X = Sup(K_1 - J_1, 0) \quad Y = Inf(K_1, J_2) \quad (4.29)$$

The sum in (4.28) is transformed in appendix B (B.31) into

$$\sum_{p=0}^{Inf(J_1, K_2)} (-)^p p! C_{J_1}^p C_{K_2}^p \frac{\Gamma(N+J-p+1)}{\Gamma(N)} \quad (4.30)$$

We can now compute the large N behaviour of $I_1(N, J_i, K_i)$; for instance, in the sector $J_1 \geq K_1, K_2 \geq J_2$ we obtain

$$I_1(N, J_i, K_i) = \delta_{J,K} J_2 N^J \left[1 + \frac{J_1(K_1 - K_2)}{2N} + \frac{1}{24N^2} F + O\left(\frac{1}{N^3}\right) \right] \quad (4.31.a)$$

$$F = J(J-1) [3 J_1^2 + J_2^2 - (J+2)] - 12 K_1 K_2 J_1 (J_1 - 1) \quad (4.31.b)$$

For the BMN behaviour, each term in (4.30) contributes, however the sum of these contributions can be performed and we obtain

$$I_1(N, J_i, K_i) \sim \delta_{J,K} \frac{2N^{J+1}}{J} sh\left(\frac{J_2 J}{2N}\right) e^{\frac{J_1(K_1 - K_2)}{2N}} \quad (4.32)$$

If we sum the graph of fig.3 and the crossed one, and if we describe each of the sectors as

$$Sup \geq A, B \geq Inf \quad (4.33)$$

with $A + B = Sup + Inf = J = K$, we obtain the large N behaviour for both graphs as

$$I(N, J_i, K_i) = \delta_{J,K} 2 Inf N^J \left[1 + \frac{1}{24N^2} F + O\left(\frac{1}{N^4}\right) \right] \quad (4.34.a)$$

$$F = J(J-1) [3 Sup^2 + Inf^2 - (J+2)] - 12 AB Sup(Sup-1) \quad (4.34.b)$$

and the BMN behaviour as

$$I(N, J_i, K_i) \sim \delta_{J,K} \frac{4N^{J+1}}{J} sh\left(\frac{J.Inf}{2N}\right) ch\left(\frac{(A-B).Sup}{2N}\right) \quad (4.35)$$

6°) Finally, we sum the contributions 4°) and 5°) to obtain the asymptotic behaviour for the complete connected four points correlation function; taking into account the relative minus sign between 4°) and 5°), we obtain

$$\Gamma_2^c(N, J_i, K_i) = -\delta_{J,K} N^{J-2} J_1 J_2 K_1 K_2 (Sup-1) + O(N^{J-4}) \quad (4.36.a)$$

$$\Gamma_2^c(N, J_i, K_i) \sim -\delta_{J,K} \frac{8N^{J+1}}{J} sh\left(\frac{J.Inf}{2N}\right) sh\left(\frac{A.Sup}{2N}\right) sh\left(\frac{B.Sup}{2N}\right) \quad (4.36.b)$$

We close this section with the following remarks. Although, in principle, we should be able to calculate the contribution of any graph, we are unable at the present time to give the expression for the sum of the graphs which enters a given k -points correlation function; this situation may improve if we use the so-called loop equations[12]. From the exemple 5°), we see that the t'Hooft topological expansion in $\frac{1}{N^2}$ is not true graph by graph and that the cancellations between graphs are far from evident. Finally, we verify on these exemples that at large N the resolvent function properly rescaled behave as

$$W_{k_1, k_2}(N, x_i \sqrt{N}, y_i \sqrt{N}) \rightarrow O\left(N^{\frac{k}{2}}\right) \quad (4.37)$$

because of the disconnected one point self closing loops (the lowest term of the expansion) and that the connected resolvent function properly rescaled behave as

$$W_{k_1, k_2}^c(N, x_i \sqrt{N}, y_i \sqrt{N}) \rightarrow O\left(\frac{1}{N^{\frac{3k-4}{2}}}\right) \quad (4.38)$$

where $k = k_1 + k_2$.

A Appendix: summation over the chains of propagators N and N^*

We consider a chain of $(p-1)$ propagators N (the arguments are the same for the chains of propagators N^*). The propagator $N_n(\xi_i, \xi_{i+1})$ has a pole at $\xi_i = \xi_{i+1}$. However, when we sum over the $p!$ chains obtained by exchanging the p points, the poles disappear as we explain now: we consider two points ξ_b and ξ_c and we look for the pole at $\xi_b = \xi_c$; many chains do not contain the propagators $N_n(\xi_b, \xi_c)$ or $N_n(\xi_c, \xi_b)$ and have no pole at $\xi_b = \xi_c$. Now, we associate by pair the chains with a propagator $N_n(\xi_b, \xi_c)$ or $N_n(\xi_c, \xi_b)$ by exchanging b and c ; the combination

$(N_n(\xi_a, \xi_b) N_n(\xi_b, \xi_c) N_n(\xi_c, \xi_d) + N_n(\xi_a, \xi_c) N_n(\xi_c, \xi_b) N_n(\xi_b, \xi_d))$ has no pole at $\xi_b = \xi_c$ and is equal to

$$\dots \left(\frac{\xi_a}{\xi_d} \right)^{n+1} \frac{\xi_d - \xi_a}{(\xi_b - \xi_a)(\xi_d - \xi_c)(\xi_c - \xi_a)(\xi_d - \xi_b)} \dots \quad (\text{A.1})$$

Similar arguments prove the absence of poles at $\xi_b = \xi_c$ if b or/and c are end-points of the chain, for instance in $(N_n(\xi_b, \xi_c) + N_n(\xi_c, \xi_b))$ or in $(N_n(\xi_a, \xi_b) N_n(\xi_b, \xi_c) + N_n(\xi_a, \xi_c) N_n(\xi_c, \xi_b))$. We remind that the self closing loops $N_n(\xi_b, \xi_b)$ are replaced in (1.26) by $H_n(\xi_b, \xi_b)$ given in (3.6.a) and contribute only to disconnected parts of the graphs ($J_i = 0$) and that the two points loop $N_n(\xi_b, \xi_c) N_n(\xi_c, \xi_b)$ is replaced in (1.26) by " $N_n(\xi_b, \xi_c) N_n(\xi_c, \xi_b)$ " which has no pole and is zero in rotationnally invariant systems.

Consequently, since we have no poles at coinciding ξ' s, the expansion in $\frac{1}{\xi_i}$ for the contribution of the $p!$ chains is the same whatever sector we choose $\xi_{\alpha_1} \leq \xi_{\alpha_2} \leq \dots \leq \xi_{\alpha_p}$ and the results must contain terms of the type $\left(\frac{1}{\xi_i}\right)^{J_i+1}$ with $J_i > 0$ if $p > 1$. According to the chosen sector we use the convergent expansion

$$\frac{1}{\xi_{i+1} - \xi_i} = \frac{1}{\xi_{i+1}} \sum_{j=0}^{\infty} \left(\frac{\xi_i}{\xi_{i+1}} \right)^j, \quad \xi_i < \xi_{i+1} \quad (\text{A.2.a})$$

$$\frac{1}{\xi_{i+1} - \xi_i} = -\frac{1}{\xi_i} \sum_{j=0}^{\infty} \left(\frac{\xi_{i+1}}{\xi_i} \right)^j, \quad \xi_i > \xi_{i+1} \quad (\text{A.2.b})$$

Now, each chain separately can be expanded in the choosen sector but if we find in the expansion some contributions $\left(\frac{1}{\xi_i}\right)^{J_i+1}$ with a negative or nul power J_i , we may ignore that contribution since we know that it must be cancelled when we sum over the $p!$ chains.

For instance, we consider three consecutive points of the chain and the corresponding situation

$$N_n(\xi_{i-1}, \xi_i) N_n(\xi_i, \xi_{i+1}), \quad \xi_i = \text{Inf}(\xi_{i-1}, \xi_i, \xi_{i+1}) \quad (\text{A.3})$$

according to (A.2) we obtain the contribution

$$-\frac{1}{\xi_{i-1} \xi_{i+1}} \left(\frac{\xi_{i-1}}{\xi_{i+1}} \right)^{n+1} \sum_{r,s=0}^{\infty} \left(\frac{\xi_i}{\xi_{i-1}} \right)^r \left(\frac{\xi_i}{\xi_{i+1}} \right)^s \quad (\text{A.4})$$

This term provides for the variable ξ_i a contribution

$$\left(\frac{1}{\xi_i} \right)^{-r-s} \rightarrow J_i = -r - s - 1 < 0 \quad (\text{A.5})$$

so that this chain should be ignored in the choosen sector. In such a sector, only the chains with no other relative minimum that the end-points of the chain should be considered.

A first important consequence of this remark is that the chains which form a loop gives no contribution; this can be verified for instance in (A.1) with $a = d$

in the case of a three points loop. A second consequence is that if we compute

$$-A_n(\eta_1^*, \xi_{b_1}) \prod_{i=1}^{p-1} N_n(\xi_{b_i}, \xi_{b_{i+1}}) K_n(\xi_{b_p}, \eta_2^*) \quad (\text{A.6})$$

in the sector $\xi_{\alpha_1} \leq \xi_{\alpha_2} \leq \dots \leq \xi_{\alpha_p}$ then , ξ_{α_1} is either ξ_{b_1} or ξ_{b_p} and along the chain we have

$$\xi_{b_1} \leq \xi_{b_2} \leq \dots \leq \xi_{b_q} = \xi_{\alpha_p} \geq \xi_{b_{q+1}} \geq \xi_{b_{q+2}} \geq \dots \geq \xi_{b_p} \quad 1 \leq q \leq p \quad (\text{A.7})$$

From (A.2), we obtain for (A.6)

$$\begin{aligned} & (-)^{p-q} \frac{1}{(\eta_1^*)^{i+1}} \left(\frac{1}{\xi_{b_1}} \right)^{i-n-r_1} \prod_{i=2}^{q-1} \left(\frac{1}{\xi_{b_i}} \right)^{r_{i-1}-r_i+1} * \\ & * \left(\frac{1}{\xi_{b_q}} \right)^{r_{q-1}+r_q+2} \prod_{i=q+1}^{p-1} \left(\frac{1}{\xi_{b_i}} \right)^{r_i-r_{i-1}+1} \left(\frac{1}{\xi_{b_p}} \right)^{n-j-r_{p-1}+1} \frac{1}{(\eta_2^*)^{-j}} \end{aligned} \quad (\text{A.8})$$

and by identification, we have

$$J_{b_1} = i - n - r_1 - 1 \quad (\text{A.9.a})$$

$$J_{b_i} = r_{i-1} - r_i \quad 2 \leq i \leq q-1 \quad (\text{A.9.b})$$

$$J_{b_q} = r_{q-1} + r_q + 1 \quad (\text{A.9.c})$$

$$J_{b_i} = r_i - r_{i-1} \quad q+1 \leq i \leq p-1 \quad (\text{A.9.d})$$

$$J_{b_p} = n - j - r_{p-1} \quad (\text{A.9.e})$$

which implies

$$J = \sum_{i=1}^p J_{b_i} = i - j \quad (\text{A.10})$$

Clearly, the variables i and r_i are all determined as function of n, j and J_{b_i} ; the constraints $i \geq n+1$ and $r_i \geq 0$ together with $J_{b_i} > 0$ implies only two constraints on j

$$j \leq n - J_+ \quad (\text{A.11.a})$$

$$j \geq n - J_+ - J_{b_q} + 1 \quad (\text{A.11.b})$$

with

$$J_+ = \sum_{i=q+1}^p J_{b_i} \quad (\text{A.12})$$

The original sum over j of the function $\frac{h_i}{h_j}$ describing the kernels $A_n(\eta_1^*, \xi_{b_1})$ and $K_n(\xi_{b_p}, \eta_2^*)$ is, for that chain, transformed into

$$\sum_{j=0}^n \frac{h_{j+J}}{h_j} \dots \Rightarrow (-)^{p-q} \sum_{j=n-J_+-J_{b_q}+1}^{n-J_+} \frac{h_{j+J}}{h_j} = (-)^{p-q} T_{J_+} \left(1 - T_{J_{a_p}}\right) \sum_{j=0}^n \frac{h_{j+J}}{h_j} \dots \quad (\text{A.13})$$

where the operator T_a is defined in (3.16).

Finally, summing over all the $p!$ chains consists in summing over all subsets

of $\{\alpha_1, \dots, \alpha_{p-1}\}$ (including the empty set and the complete set) corresponding to the definition of J_+ . Taking into account the alternate sign in $(-)^{p-q}$ the total sum is nothing but

$$\prod_{i=1}^p (1 - T_{J_i}) \sum_{j=0}^n \frac{h_{j+J}}{h_j} \dots \quad (\text{A.14})$$

This result is the purpose of this appendix.

B Appendix: the 't Hooft expansion

We define the symbol

$$n^{*J} = (n+1)(n+2) \dots (n+J) \quad (\text{B.1})$$

which is expanded in powers of n as

$$n^{*J} = \sum_{p=0}^J \sigma_p(J) n^{J-p} \quad (\text{B.2})$$

with

$$\sigma_0(J) = 1 \quad \sigma_J(J) = J! \quad (\text{B.3.a})$$

$$\sigma_1(J) = \frac{J(J+1)}{2} \quad \sigma_p(J) = 0 \quad \text{if } p > J \quad (\text{B.3.b})$$

For a given p , the coefficients $\sigma_p(J)$ are polynomials in J ; clearly we have

$$\sigma_p(J) = \sum_{1 \leq k_1 < k_2 < \dots < k_p \leq J} k_1 k_2 \dots k_p \quad (\text{B.4})$$

and we observe the relation

$$\sigma_p(J) = \sigma_p(J-1) + J \sigma_{p-1}(J-1) \quad (\text{B.5})$$

Since $\sigma_p(J)$ vanishes if $J < p$, we write

$$\sigma_p(J) = \frac{1}{2^p p!} J(J-1) \dots (J-p+1) \pi_p(J) \quad (\text{B.6.a})$$

$$\pi_0(J) = 1 \quad \pi_1(J) = J+1 \quad (\text{B.6.b})$$

The relation

$$J \pi_p(J) = (J-p) \pi_p(J-1) + 2pJ \pi_{p-1}(J-1) \quad (\text{B.7})$$

shows that $\pi_p(J)$ is a monic polynomial of degree p in J which vanishes at $J = -1$ if $p > 0$. We find

$$\pi_2(J) = (J+1) \left(J + \frac{2}{3} \right) \quad \pi_3(J) = J(J+1)^2 \quad (\text{B.8.a})$$

$$\pi_4(J) = (J+1) \left(J^3 + J^2 - \frac{2}{3}J - \frac{8}{15} \right) \quad (\text{B.8.b})$$

$$\pi_5(J) = J(J+1)^2 \left(J^2 - \frac{1}{3}J - 2 \right) \quad (\text{B.8.c})$$

We note that at large J

$$\sigma_p(J) \sim \frac{1}{p!} \left(\frac{J^2}{2} \right)^p \quad (\text{B.9})$$

so that the so called "BMN" limit (n and $J \rightarrow \infty$, $\frac{J^2}{n} = cte$) is

$$n^{*J} \sim n^J e^{\frac{J^2}{2n}} \quad (\text{B.10})$$

From the generating functional

$$(1+x)^{-j-1} = \sum_{J=0}^{\infty} (-)^J j^{*J} \frac{x^J}{J!} \quad (\text{B.11})$$

and from

$$\sum_{j=0}^n (1+x)^{-j-1} = \frac{1 - (1+x)^{-n-1}}{x} \quad (\text{B.12})$$

we obtain by identification of the powers of x

$$\sum_{j=0}^n j^{*J} = \frac{n^{*(J+1)}}{J+1} \quad (\text{B.13})$$

In section 4 we have to calculate

$$\sum_{j=N-J}^{N-1} j^{*J} = \frac{(N-1)^{*(J+1)} - (N-J-1)^{*(J+1)}}{J+1} \quad (\text{B.14})$$

From

$$(N-1)^{*(J+1)} = N(N+1) \dots (N+J) = \sum_{p=0}^J \sigma_p(J) N^{J-p+1} \quad (\text{B.15.a})$$

$$(N-J-1)^{*(J+1)} = N(N-1) \dots (N-J) = \sum_{p=0}^J (-)^p \sigma_p(J) N^{J-p+1} \quad (\text{B.15.b})$$

we obtain the topological t'Hooft expansion

$$\sum_{j=N-J}^{N-1} j^{*J} = \frac{2}{J+1} \sum_{p \text{ odd}}^J \sigma_p(J) N^{J-p+1} \quad (\text{B.16})$$

Also, from (B.9), the BMN limit (large N , large J , $\frac{J^2}{N} = cte$) is

$$\sum_{j=N-J}^{N-1} j^{*J} \sim \frac{2N^{J+1}}{J} sh\left(\frac{J^2}{2N}\right) \quad (\text{B.17})$$

We find convenient to introduce the expression

$$n^{** (J+1)} = e^{-\left(\frac{J}{2}+1\right) \frac{\partial}{\partial n}} n^{*(J+1)} \quad (\text{B.18.a})$$

$$n^{** (J+1)} = \left(n - \frac{J}{2}\right) \left(n - \frac{J}{2} + 1\right) \dots \left(n + \frac{J}{2} - 1\right) \left(n + \frac{J}{2}\right) \quad (\text{B.18.b})$$

which has clearly a $\frac{1}{n^2}$ expansion

$$n^{** (J+1)} = \sum_{p=0}^{E\left(\frac{J+1}{2}\right)} (-)^p \Sigma_{2p}(J) n^{J-2p+1} \quad (\text{B.19})$$

with

$$\Sigma_0(J) = 1 \quad \Sigma_2(J) = \frac{J(J+1)(J+2)}{24} \quad (\text{B.20})$$

We have the relation

$$\Sigma_{2p}(J) = \Sigma_{2p}(J-2) + \frac{J^2}{4} \Sigma_{2p-2}(J-2) \quad (\text{B.21})$$

We write for $p > 0$

$$\Sigma_{2p}(J) = \frac{(J-2p+2)(J-2p+3)\dots J(J+1)}{(24)^p p!} \theta_{2p}(J) \quad (\text{B.22.a})$$

$$\theta_0(J) = 1 \quad \theta_2(J) = J+2 \quad (\text{B.22.b})$$

the relation (B.21) becomes

$$J(J+1) \theta_{2p}(J) = (J-2p)(J-2p+1) \theta_{2p}(J-2) + 6pJ^2 \theta_{2p-2}(J-2) \quad (\text{B.23})$$

The polynomials $\theta_{2p}(J)$ are monic polynomials of degree p in J which vanish for $J = -2$ if $p > 0$. We have

$$\theta_4(J) = (J+2) \left(J + \frac{12}{5} \right) \quad (\text{B.24.a})$$

$$\theta_6(J) = (J+2) \left(J^2 + \frac{26}{5}J + \frac{48}{7} \right) \quad (\text{B.24.b})$$

$$\theta_8(J) = (J+2) \left(J^3 + \frac{42}{5}J^2 + \frac{4184}{175}J + \frac{576}{25} \right) \quad (\text{B.24.c})$$

At large J

$$\Sigma_{2p}(J) \sim \frac{1}{p!} \left(\frac{J^3}{24} \right)^p \quad (\text{B.25})$$

so that the "BMN" limit (and the first correction) of $n^{*(J+1)}$ is

$$n^{*(J+1)} \sim n^{J+1} e^{-\frac{J^3}{24n^2}} = n^{J+1} \left[1 + O\left(\frac{1}{J}\right) \right], \quad \frac{J^2}{n} = cte \quad (\text{B.26})$$

the exponential term being the first correction to the "BMN" limit.

With these notations, we may compute directly the $\frac{1}{N^2}$ expansion of

$$\prod_{i=1}^p (1 - T_{J_i}) \sum_{k=0}^{N-1} \frac{\Gamma(k+J+1)}{\Gamma(k+1)} = \frac{2^p}{J+1} \prod_{i=1}^p \left[sh \left(\frac{J_i}{2} \frac{\partial}{\partial N} \right) \right] N^{*(J+1)} \quad (\text{B.27})$$

where $\sum_{i=1}^p J_i = J$; the "BMN" limit is trivially

$$\prod_{i=1}^p (1 - T_{J_i}) \sum_{k=0}^{N-1} \frac{\Gamma(k+J+1)}{\Gamma(k+1)} \sim 2^p \frac{N^{J+1}}{J} \prod_{i=1}^p \left[sh \left(\frac{J_i J}{2N} \right) \right], \quad \frac{J_i J}{N} = cte \quad (\text{B.28})$$

Finally, if we compare the coefficients of the term x^b in the expansions

$$(1+x)^{-j-1} = (1+x)^{-j-a-1} (1+x)^a \quad (\text{B.29})$$

we obtain the relation

$$\frac{\Gamma(j+b+1)}{\Gamma(j+1)} = \sum_{q=0}^{Inf(a,b)} (-)^q q! C_a^q C_b^q \frac{\Gamma(j+a+b-q+1)}{\Gamma(j+a+1)} \quad (\text{B.30})$$

if a is a non negative integer (with trivial analytic continuation to any a). As a consequence

$$\begin{aligned} & \sum_{j=0}^{N-1} \frac{\Gamma(j+a+c+1)}{\Gamma(j+1)} \frac{\Gamma(j+b+c+1)}{\Gamma(j+c+1)} \\ &= \sum_{q=0}^{Inf(a,b)} (-)^q \frac{q! C_a^q C_b^q}{(a+b+c-q+1)} \frac{\Gamma(N+a+b+c-q+1)}{\Gamma(N)} \end{aligned} \quad (\text{B.31})$$

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